§ 9.12

Cor 9.12

a) Let A be a symmetrizable matrix and let $\alpha \in \Delta_+(A)$ be such that $(\alpha|\alpha) \neq 0$. Then $\bigoplus_{k\geq 1} g_{k\alpha}$ is a free Lie algebra on a basis of the space $\bigoplus_{k\geq 1} g^{\circ}_{k\alpha}$, where $g^{\circ}_{k\alpha} = \{x \in g_{k\alpha} \mid (x|y)=0$ for all y from the subalgebra generated by $g_{-\alpha}, \cdots, g_{-(k-1)\alpha}\}$.

proof: follows from prop 9.12 by setting $L = \{k\alpha \mid k \in \mathbb{Z}_+\}$.

prop 5.5 : $\alpha \in \Delta_+^{im}$, $n\alpha \in \Delta_+^{im}$ if $n \in \mathbb{Z}_+$ $\alpha \in \Delta_+$

$g^{\circ}_{k\alpha} = \{x \in g_{k\alpha} \mid (x|y)=0$ for all $y \in [n_-^L, n_-^L]\}$.

$(g_\alpha | g_{-\alpha}) \neq 0$.

b) Let A be a generalized Cartan matrix and let $\alpha \in \Delta_+(A)$ be an isotropic root. Then $\mathbb{C}\nu^{-1}(\alpha) \oplus (\bigoplus_{j\neq 0} g_{j\alpha})$ is an infinite Heisenberg Lie algebra.

proof: if α is an imaginary root of an affine Lie algebra. then b) amounts to prop 8.4.

Recall : prop 8.4 : Let $g(A)$ be an affine algebra.

a) Set $t = \mathbb{C}K + \sum_{s\in\mathbb{Z}} g_{s\delta}$, Then t is isomorphic to the infinite-dimen. Heisenberg alg. with center $\mathbb{C}K$.

Applying prop 5.7 and lem 3.8 prove b) in the general case.

Recall : prop 5.7 : Let A be symmetrizable, A root α is isotropic (i.e: $(\alpha|\alpha)=0$) iff it is W-equivalent to an imaginary root β such that supp β is a subdiagram of affine type of $S(A)$ (then $\beta = k\delta$)

lem 3.8. $r_i^{ad} \in \mathrm{Aut}\, g(A)$ & $r_i^{ad}|_{\mathfrak{H}} = r_i$

=

§ 9.13.

· Heisenberg Lie algebra of order ∞. That is a Lie algebra with a basis p_i, q_i $(i = 1, 2, \cdots)$ and c, with the following commutation relations:

$[p_i, q_i] = c$ $(i = 1, 2, \cdots)$ and all the other brackets are zero. This is a nilpotent Lie algebra with center $\mathbb{C}c$.

· It is well known that for every $a \in \mathbb{C}^\times$, the Lie algebra s has an irreducible representation σ_a, called canonical commutation relation representation, on the space $R = \mathbb{C}[x_1, x_2, \cdots]$ of polynomials in infinitely many indeterminates x_j, defined by:

$$\sigma_a(p_i) = a\frac{\partial}{\partial x_i}, \quad \sigma_a(q_i) = x_i, \quad \sigma_a(c) = a I_R.$$

Weyl algebra. Schrödinger representation.

We denote this s-module by R_a.

· Introduce the following commutative subalgebras of s.

$$s_+ = \sum_{j \geq 1} \mathbb{C} p_j, \quad s_- = \sum_{j \geq 1} \mathbb{C} q_j.$$

· A vector v of an s-module is called a vacuum vector with eigenvalue $\lambda \in \mathbb{C}$ if $s_+(v) = 0$ and $c(v) = \lambda v$.

· Note that s can be viewed as the Lie algebra $g'(0)/t_1$, where $t_1 = \sum \mathbb{C}(\alpha_i^\vee - \alpha_j^\vee) \subset t$ is a central ideal, so that n_+ (resp. n_-) is identified with s_+ (resp. s_-).

· Since (R, σ_a) is a free $U(s_-)$-module of rank 1, it is a Verma module over s, the vector 1 being the highest weight vector = vacuum vector.

Stone – von Neumann theorem:

the canonical commutation relations on two generators (canonical coordinate q and canonical momentum p) in the form $[q, p] = i\hbar$ may be represented as unbounded operators on the Hilbert space of square integrable function $L^2(\mathbb{R})$ on the real line by defining them on the dense subspace of smooth function $\psi : \mathbb{R} \to \mathbb{C}$ as

$$(q\psi)(x) := x\psi(x) \qquad (p\psi)(x) := -i\hbar\frac{\partial}{\partial x}\psi(x).$$

where on the right we have the derivative along the canonical coordinate function on $\mathbb{R}$. Schrödinger rep.

Cor. 9.13 let V be an irreducible s-module which has a nonzero vaccum vector with a nonzero eigenvalue λ, Then the s-module V is isomorphic to R_λ.

Lem 9.13.

a) let V be an s-module such that $c = a I_V$, where $a \neq 0$. which has a vacuum vector $v_0 \neq 0$, such that $V = U(s_-)(v_0)$. Then the s-module V is isomorphic to R_a.

b) let V be an s-module such that c is diagonalizable with nonzero eigenvalues and such that for every $v \in V$ there exists N such that $p_{i_1} \cdots p_{i_n}(v) = 0$, whenever $n > N$, then V is isomorphic to a direct sum of s-modules of the form R_a, $a \neq 0$.

p_i $\qquad$ $p_i(v) = 0.$

$g'(0)/\varepsilon_1$

proof: we can assume in b) that $c = a I_V$ with $a \neq 0$.

V may be viewed as a $g'(0)$-module for which $\alpha_i^\vee = a I_V$ for all i. But then for every weight λ and for $\beta \in Q$ we have

$\langle \lambda, \nu^{-1}(\beta) \rangle = a\,\mathrm{ht}\,\beta$. $\quad (\beta = \sum_i k_i \alpha_i)$.

since $(\beta|\beta) = 0$ and $\rho = 0$. $\quad$ (Lie algebra $g(0) \longrightarrow n \times n$ zero matrix (including $n = \infty$))

$\langle \rho, \alpha_i^\vee \rangle = \frac{1}{2} a_{ii} = 0$.

we have $\underline{2\langle \lambda + \rho, \nu^{-1}(\beta) \rangle = 2a\,\mathrm{ht}(\beta) \neq (\beta|\beta)}$ for $\beta \in Q_+ \setminus \{0\}$ (*)

by prop 9.10 a) and b).

· The Lie algebra s is often extended by a derivation d_0 defined by: $[d_0, q_j] = m_j q_j$, $\quad [d_0, p_j] = -m_j p_j$.

where m_j are some positive integers.

The Lie algebra $\mathcal{A} = (s + \mathbb{C} d_0) \oplus a_0$, where a_0 is a finite-dimensional central ideal, is called an oscillator algebra.

· Given $b \in \mathbb{C}$ and $\lambda \in a_0^*$, we can extend the s-module R_a to the $\mathcal{A}$-module $R_{a,b,\lambda}$ as follows:

$d_0 \longmapsto b + \sum_j m_j x_j \frac{\partial}{\partial x_j}$, $\quad a \longmapsto \langle \lambda, a \rangle 1$ for $a \in a_0$.

let $s_0 = \mathbb{C}c + \mathbb{C}d_0 + a_0$, we have the trianglur decomposition

$$\mathcal{A} = s_- \oplus s_0 \oplus s_+.$$

prop 9.13. let V be an $\mathcal{A}$-module such that s_0 is diagonalizable and c has only nonzero eigenvalues:

a) If there exists $v_0 \in V$, $v_0 \neq 0$, such that

$$s_+(v_0) = 0, \quad U(s_-) v_0 = V.$$

Then V is isomorphic to an $\mathcal{A}$-module $R_{a,b,\lambda}$.

b) If for every $v \in V$, there exists N such that $p_{i_1} \cdots p_{i_n}(v) = 0$ whenever $n > N$, then V is isomorphic to a direct sum of $\mathcal{A}$-modules $R_{a,b,\lambda}$, $a \neq 0$.

· Note that the monomial $x_1^{j_1} \cdots x_n^{j_n} \in R_{a,b,\lambda}$ is an eigenvector of d_0 with eigenvalue $\sum_k m_k j_k + b$.

$$d_0(x_1^{j_1} \cdots x_n^{j_n}) = b x_1^{j_1} \cdots x_n^{j_n} + \sum_k m_k j_k x_1^{j_1} \cdots x_n^{j_n} = (\sum_k m_k j_k + b)(x_1^{j_1} \cdots x_n^{j_n})$$

· Hence, for the $\mathcal{A}$-module $R = R_{a,b,\lambda}$ with $a \neq 0$, we have

$$\underline{\mathrm{tr}_R\, q^{d_0} = q^b \prod_{j=1}^{\infty} (1 - q^{m_j})^{-1}}$$

$$\Gamma\ \mathrm{tr}_R\, q^{d_0} = \sum q^{\lambda} = \sum q^{b + \sum_k m_k j_k} = q^b \sum q^{\sum_k m_k j_k} = q^b \prod_{m_k} \sum_{j_k} q^{j_k m_k}$$

$$= q^b \prod_{m_k} (q^0 + q^{m_k} + q^{2m_k} + \cdots) = q^b \prod_{m_k} (1 - q^{m_k})^{-1} \quad \lrcorner$$

本質刻画. character.

Here, as usual, for a diagonalizable operator A on a vector space V with eigenvalues $\lambda_1, \lambda_2, \cdots$ counting the multi. one define: $\mathrm{tr}_V\, q^A = \sum_j q^{\lambda_j}$.

· $B(p_n x, y) = B(x, q_n y) \Rightarrow p_n$ is adjoint to q_n.

by (19.4.2) $B(g(x), y) = -B(x, w(g)(y))$

$$\Rightarrow B(e_i x, y) = B(x, f_i y).$$

& $B(v_\lambda, v_\lambda) = \langle v_\lambda \rangle = 1$

· claim: distinct monomials are orthogonal w.r.t. B and that: $\underline{B(x_1^{k_1} \cdots x_n^{k_n}, x_1^{k_1} \cdots x_n^{k_n}) = a^{\sum k_j} \prod_j k_j!}$

proof: $B(a v_\lambda, a' v_\lambda) = \langle \hat{w}(a) a' v_\lambda \rangle$ where expectation value $\langle v \rangle \in \mathbb{C}$. satisfies $v = \underline{\langle v \rangle v_\lambda} + \sum_{\alpha \in \mathcal{Q}\setminus\{0\}} v_{\lambda-\alpha}$, where $v_{\lambda-\alpha} \in V_{\lambda-\alpha}$.

then $B(x_1, x_1) = B(q_1 \cdot 1, q_1 \cdot 1) = \underline{\langle p_1 q_1 \cdot 1 \rangle} = a$.

$$B(x_1^{k_1}, x_1^{k_1}) = B(q_1^{k_1}.1,\ q_1^{k_1}.1) = a^{k_1} k_1!$$

$$B(x_1, x_2) = B(q_1.1,\ q_2.1) = \langle p_1 q_2.1 \rangle = 0$$

· As in §9.4. B can be written also in the following form:

$$B(P, Q) = \left(P\left(a\tfrac{\partial}{\partial x_1}, a\tfrac{\partial}{\partial x_2}, \cdots\right) Q(x_1, x_2, \cdots)\right)(0).$$

§ 9.14.

Recall: The Lie algebra $\mathfrak{d} := \bigoplus_{j\in\mathbb{Z}} \mathbb{C} d_j$ has a unique (up to isomorphism) nontrivial central extension by a 1-dimensional center, say $\mathbb{C}c$, called the Virasoro algebra Vir, which is defined by the following comm. relation:

$$[d_i, d_j] = (i-j) d_{i+j} + \tfrac{1}{12}(i^3 - i)\delta_{i,-j}\, c \qquad (i, j \in \mathbb{Z}).$$

· Define the triangular decomposition of Vir as follows:

$$Vir = Vir_- \oplus Vir_0 \oplus Vir_+,$$

where $Vir_\pm = \bigoplus_{j>0} \mathbb{C} d_{\pm j}$, $Vir_0 = \mathbb{C}c \oplus \mathbb{C} d_0$.

· Given $c, h \in \mathbb{C}$, define a Vir-module V with highest weight (c, h) by the requirement that there exists a nonzero vector $v = v_{c,h}$, s.t.

$$Vir_+(v) = 0, \quad \underline{U(Vir_-)v = V}, \quad d_0(v) = hv, \quad c(v) = cv.$$

· It is clear that c acts on $M(c,h)$ as cI.

$$(cU(Vir_-)v = U(Vir_-)c.v).$$

The number c is called the conformal central charge.

· As in §9.7 we easily show that the elements

(9.14.1) $d_{-j_n} \cdots d_{-j_2} d_{-j_1}(v_{c,h})$ where $0 < j_1 \leq j_2 \leq \cdots$

form a basis of $M(c,h)$. Since $[d_0, d_{-n}] = n d_{-n}$,

we see that d_0 is diagonalizable on $M(c,h)$ with spectrum $h+\mathbb{Z}_+$ and with the eigenspace decomposition

$$M(c,h) = \bigoplus_{j\in\mathbb{Z}_+} M(c,h)_{h+j}.$$

where $M(c,h)_{h+j}$ is spanned by elements of the form (9.14.1) with $j_1+\cdots+j_n=j$.

- It follows that $\dim M(c,h)_{h+j} = P(j)$ where $P(j)$ is the classical partition function.
- In number theory, the partition function $P(n)$ represents the number of possible partitions of a nonnegative integer n. $p(0)=1$, $p(n)=0$, $n<0$.
- In representation theory, the Kostant partition function of a root system Δ is the number of ways one can represent a vector or weight as a non-negative integer linear combination of the positive roots $\Delta^+ \subset \Delta$
- Equation (9.14.3) can be rewritten as follows:

$$tr_{M(c,h)}\, q^{d_0} := \sum_\lambda \dim M(c,h)_\lambda\, q^\lambda = q^h \prod_{j=1}^{\infty} (1-q^j)^{-1}$$

- As in §9.7, the series $tr_V\, q^{d_0}$ is called the formal character of Vir-module V.
- The Chevalley involution ω of Vir is defined by

$$\omega(d_n) = -d_{-n}, \quad \omega(c) = -c.$$